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Author(s)	Komatsu, Takao
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# Convolution identities for Cauchy numbers of the first kind and of the second kind

Takao Komatsu <sup>1</sup>

School of Mathematics and Statistics, Wuhan University

## 1 Introduction

The Cauchy numbers  $c_n$  ( $n \geq 0$ ) are defined by

$$c_n = \int_0^1 x(x-1)\dots(x-n+1)dx$$

and the generating function of  $c_n$  is given by

$$\frac{x}{\ln(1+x)} = \sum_{n=0}^{\infty} c_n \frac{x^n}{n!} \quad (|x| < 1)$$

([4, 11]). Several initial values are

$$c_0 = 1, c_1 = \frac{1}{2}, c_2 = -\frac{1}{6}, c_3 = \frac{1}{4}, c_4 = -\frac{19}{30}, c_5 = \frac{9}{4}, c_6 = -\frac{863}{84}, c_7 = \frac{1375}{24}.$$

## 2 Preliminaries

$c(x) = x/\ln(1+x)$  satisfies the identity

$$c(x)^2 = (1+x)c(x) - (1+x)xc'(x). \quad (1)$$

Since for  $i, \nu \geq 0$  it holds that

$$x^i c^{(\nu)}(x) = \sum_{n=0}^{\infty} \frac{n!}{(n-i)!} c_{n+\nu-i} \frac{x^n}{n!}, \quad (2)$$

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the identity (1) immediately leads to the formula

$$\sum_{k=0}^n \binom{n}{k} c_k c_{n-k} = -n(n-2)c_{n-1} - (n-1)c_n \quad (n \geq 0). \quad (3)$$

Differentiating both sides of (1) by  $x$  and dividing them by 2, we obtain

$$c(x)c'(x) = -\frac{1}{2}x(x+1)c''(x) - \frac{1}{2}xc'(x) + \frac{1}{2}c(x). \quad (4)$$

**Proposition 1.**

$$c(x)^3 = \frac{1}{2}(x+1)(x+2)c(x) - \frac{1}{2}x(x+1)(x+2)c'(x) + \frac{1}{2}x^2(x+1)^2c''(x). \quad (5)$$

*Proof.* By (1) and (4),

$$\begin{aligned} c(x)^3 &= (1+x)((1+x)c(x) - (1+x)xc'(x)) \\ &\quad - (1+x)x\left(-\frac{1}{2}x(x+1)c''(x) - \frac{1}{2}xc'(x) + \frac{1}{2}c(x)\right) \\ &= \frac{1}{2}(x+1)(x+2)c(x) - \frac{1}{2}x(x+1)(x+2)c'(x) + \frac{1}{2}x^2(x+1)^2c''(x). \end{aligned}$$

□

**Theorem 1.** For  $n \geq 2$  we have

$$\begin{aligned} &\sum_{\substack{k_1+k_2+k_3=n \\ k_1, k_2, k_3 \geq 0}} \frac{n!}{k_1!k_2!k_3!} c_{k_1} c_{k_2} c_{k_3} \\ &= \frac{(n-1)(n-2)}{2} c_n + \frac{n(n-2)(2n-5)}{2} c_{n-1} + \frac{n(n-1)(n-3)^2}{2} c_{n-2}. \end{aligned}$$

*Remark.* In [2, Corollary 3]

$$\sum_{\substack{k_1+k_2+k_3=n \\ k_1, k_2, k_3 \geq 0}} \frac{n!}{k_1!k_2!k_3!} B_{k_1} B_{k_2} B_{k_3} = \frac{(n-1)(n-2)}{2} B_n + \frac{3n(n-2)}{2} B_{n-1} + n(n-1) B_{n-2}.$$

*Proof of Theorem 1.* By (2) in Proposition 1

$$\begin{aligned}
& \frac{1}{2}(x+1)(x+2)c(x) - \frac{1}{2}x(x+1)(x+2)c'(x) + \frac{1}{2}x^2(x+1)^2c''(x) \\
&= \sum_{n=0}^{\infty} \left( c_n + \frac{3}{2}nc_{n-1} + \frac{1}{2}n(n-1)c_{n-2} \right) \frac{x^n}{n!} \\
&\quad - \sum_{n=0}^{\infty} \left( nc_n + \frac{3}{2}n(n-1)c_{n-1} + \frac{1}{2}n(n-1)(n-2)c_{n-2} \right) \frac{x^n}{n!} \\
&\quad + \sum_{n=0}^{\infty} \left( \frac{1}{2}n(n-1)c_n + n(n-1)(n-2)c_{n-1} + \frac{1}{2}n(n-1)(n-2)(n-3)c_{n-2} \right) \frac{x^n}{n!} \\
&= \sum_{n=0}^{\infty} \left( \frac{(n-1)(n-2)}{2}c_n + \frac{n(n-2)(2n-5)}{2}c_{n-1} + \frac{n(n-1)(n-3)^2}{2}c_{n-2} \right) \frac{x^n}{n!}.
\end{aligned}$$

□

The fundamental result of the third order is given by the following.

**Theorem 2.** For  $\mu, n \geq 0$ , we have

$$\begin{aligned}
& \sum_{\substack{\kappa_1+\kappa_2+\kappa_3=\mu \\ \kappa_1, \kappa_2, \kappa_3 \geq 0}} \frac{\mu!}{\kappa_1!\kappa_2!\kappa_3!} (c_{\kappa_1} + c_{\kappa_2} + c_{\kappa_3})^n = \frac{(n+\mu-1)(n+\mu-2)}{2} c_{n+\mu} \\
& \quad + \frac{(n+\mu)(n+\mu-2)(2n+2\mu-5)}{2} c_{n+\mu-1} \\
& \quad + \frac{(n+\mu)(n+\mu-1)(n+\mu-3)^2}{2} c_{n+\mu-2}.
\end{aligned}$$

*Remark.* If we put  $\mu = 0$ , we have the identity in Theorem (1). If we put  $\mu = 1$ , we have

$$\begin{aligned}
& (c_0 + c_0 + c_1)^n \\
&= \frac{n(n-1)}{6} c_{n+1} + \frac{(n+1)(n-1)(2n-3)}{6} c_n + \frac{n(n+1)(n-2)^2}{6} c_{n-1}.
\end{aligned}$$

If we put  $\mu = 2$ , we have

$$\begin{aligned}
& (c_0 + c_0 + c_2)^n + 2(c_0 + c_1 + c_1)^n \\
&= \frac{n(n+1)}{6} c_{n+2} + \frac{n(n+2)(2n-1)}{6} c_{n+1} + \frac{(n+1)(n+2)(n-1)^2}{6} c_n.
\end{aligned}$$

If we put  $\mu = 3$ , we have

$$\begin{aligned} & (c_0 + c_0 + c_3)^n + 6(c_0 + c_1 + c_2)^n + 2(c_1 + c_1 + c_1)^n \\ &= \frac{(n+1)(n+2)}{6}c_{n+3} + \frac{(n+1)(n+3)(2n+1)}{6}c_{n+2} + \frac{n^2(n+2)(n+3)}{6}c_{n+1}. \end{aligned}$$

To prove Theorem 2 is based upon a relation about the function  $c(x)$ .

**Proposition 2.** For  $\mu \geq 0$ , we have

$$\begin{aligned} & \sum_{\substack{\kappa_1 + \kappa_2 + \kappa_3 = \mu \\ \kappa_1, \kappa_2, \kappa_3 \geq 0}} \frac{\mu!}{\kappa_1! \kappa_2! \kappa_3!} c^{(\kappa_1)}(x) c^{(\kappa_2)}(x) c^{(\kappa_3)}(x) \\ &= \frac{1}{2} x^2 (x+1)^2 c^{(\mu+2)}(x) \\ & \quad + \frac{1}{2} x(x+1)((4\mu-1)x + (2\mu-2)) c^{(\mu+1)}(x) \\ & \quad + \frac{1}{2} ((6\mu^2 - 9\mu + 1)x^2 + 3(2\mu^2 - 4\mu + 1)x + (\mu-1)(\mu-2)) c^{(\mu)}(x) \\ & \quad + \frac{\mu}{2} ((4\mu^2 - 15\mu + 13)x + (2\mu-5)(\mu-2)) c^{(\mu-1)}(x) \\ & \quad + \frac{1}{2} \mu(\mu-1)(\mu-3)^2 c^{(\mu-2)}(x). \end{aligned}$$

*Proof.* By differentiating both sides of (5)  $\mu$  times with respect to  $x$ , we have the desired result. The left-hand side is due to the General Leibniz's rule. The right-hand side can be proved by induction.  $\square$

*Proof of Theorem 2.* By (2) in Proposition 2, we have

$$\begin{aligned} & \frac{1}{2} x^2 (x+1)^2 c^{(\mu+2)}(x) \\ &= \sum_{n=0}^{\infty} \left( \frac{1}{2} n(n-1) c_{n+\mu} + n(n-1)(n-2) c_{n+\mu-1} + \frac{1}{2} n(n-1)(n-2)(n-3) c_{n+\mu-2} \right) \frac{x^n}{n!}, \\ & \frac{1}{2} x(x+1)((4\mu-1)x + (2\mu-2)) c^{(\mu+1)}(x) \\ &= \sum_{n=0}^{\infty} \left( (\mu-1) n c_{n+\mu} + \frac{6\mu-3}{2} n(n-1) c_{n+\mu-1} + \frac{4\mu-1}{2} n(n-1)(n-2) c_{n+\mu-2} \right) \frac{x^n}{n!}, \end{aligned}$$

$$\begin{aligned} & \frac{1}{2}((6\mu^2 - 9\mu + 1)x^2 + 3(2\mu^2 - 4\mu + 1)x + (\mu - 1)(\mu - 2))c^{(\mu)}(x) \\ &= \sum_{n=0}^{\infty} \left( \frac{(\mu - 1)(\mu - 2)}{2} c_{n+\mu} + \frac{3(2\mu^2 - 4\mu + 1)}{2} n c_{n+\mu-1} + \frac{6\mu^2 - 9\mu + 1}{2} n(n-1) c_{n+\mu-2} \right) \frac{x^n}{n!}, \end{aligned}$$

$$\begin{aligned} & \frac{\mu}{2}((4\mu^2 - 15\mu + 13)x + (2\mu - 5)(\mu - 2))c^{(\mu-1)}(x) \\ &= \sum_{n=0}^{\infty} \left( \frac{\mu(2\mu - 5)(\mu - 2)}{2} c_{n+\mu-1} + \frac{\mu(4\mu^2 - 15\mu + 13)}{2} n c_{n+\mu-2} \right) \frac{x^n}{n!} \end{aligned}$$

and

$$\frac{1}{2}\mu(\mu - 1)(\mu - 3)^2 c^{(\mu-2)}(x) = \sum_{n=0}^{\infty} \frac{\mu(\mu - 1)(\mu - 3)^2}{2} c_{n+\mu-2} \frac{x^n}{n!}.$$

Combining all the relations together, we obtain the desired result.  $\square$

### 3 Higher powers

In similar manners to Proposition 1, we have the following.

$$\begin{aligned} c(x)^4 &= \frac{(1+x)(x^2 + 6x + 6)}{6} c(x) - \frac{x(1+x)(x^2 + 6x + 6)}{6} c'(x) \\ &\quad + \frac{x^2(1+x)^2}{2} c''(x) - \frac{x^3(1+x)^3}{6} c'''(x), \\ c(x)^5 &= \frac{(1+x)(x^3 + 14x^2 + 36x + 24)}{24} c(x) - \frac{x(1+x)(x^3 + 14x^2 + 36x + 24)}{24} c'(x) \\ &\quad + \frac{x^2(1+x)^2(x^2 + 6x + 12)}{24} c''(x) + \frac{(x-2)x^3(1+x)^3}{12} c^{(3)}(x) \\ &\quad + \frac{x^4(1+x)^4}{24} c^{(4)}(x), \end{aligned}$$

Therefore,

$$\begin{aligned}
& (c_0 + c_0 + c_0 + c_0)^n \\
&= -\frac{(n-1)(n-2)(n-3)}{6}c_n - \frac{n(n-2)(n-3)^2}{2}c_{n-1} \\
&\quad - \frac{n(n-1)(n-3)(3n^2-21n+37)}{6}c_{n-2} - \frac{n(n-1)(n-2)(n-4)^3}{6}c_{n-3}, \\
& (c_0 + c_0 + c_0 + c_0 + c_0)^n \\
&= \frac{(n-1)(n-2)(n-3)(n-4)}{24}c_n + \frac{n(n-2)(n-3)(n-4)(2n-7)}{12}c_{n-1} \\
&\quad + \frac{n(n-1)(n-3)(n-4)(6n^2-48n+97)}{24}c_{n-2} \\
&\quad + \frac{n(n-1)(n-2)(n-4)(2n-9)(2n^2-18n+41)}{24}c_{n-3} \\
&\quad + \frac{n(n-1)(n-2)(n-3)(n-5)^4}{24}c_{n-4}.
\end{aligned}$$

In general, we have the following.

**Theorem 3.** For any integers  $n \geq 1$  and  $m \geq 2$ , we have

$$\begin{aligned}
& (\underbrace{c_0 + \dots + c_0}_m)^n \\
&= \frac{n!}{(m-1)!} \sum_{i=0}^{m-1} \left( \sum_{l=0}^{m-1} \sum_{k=0}^{m-l-1} \frac{(-1)^{l+k}(m-k-1)!}{l!(n-l-i)!} \left\{ \begin{matrix} m-1 \\ m-k-1 \end{matrix} \right\} \binom{m-k-1}{i-k+1} \right) c_{n-i}.
\end{aligned}$$

By applying Theorem 3, we have the following result.

**Theorem 4.** For any integers  $n \geq 1$  and  $m \geq 2$ , we have

$$\begin{aligned}
& \sum_{\substack{\kappa_1 + \dots + \kappa_m = \mu \\ \kappa_1, \dots, \kappa_m \geq 0}} \frac{\mu!}{\kappa_1! \dots \kappa_m!} (c_{\kappa_1} + \dots + c_{\kappa_m})^n \\
&= \frac{(n+\mu)!}{(m-1)!} \sum_{i=0}^{m-1} \left( \sum_{l=0}^{m-1} \sum_{k=0}^{m-l-1} \frac{(-1)^{l+k}(m-k-1)!}{l!(n-l-i)!} \left\{ \begin{matrix} m-1 \\ m-k-1 \end{matrix} \right\} \binom{m-k-1}{i-k+1} \right) c_{n+\mu-i}.
\end{aligned}$$

When  $\mu = 1$  in Theorem 4, we have for any integers  $n \geq 1$  and  $m \geq 2$ ,

$$\begin{aligned} & \underbrace{(c_0 + \cdots + c_0 + c_1)}_{m-1}^n \\ &= \frac{(n+1)!}{m!} \sum_{i=0}^{m-1} \left( \sum_{l=0}^{m-1} \sum_{k=0}^{m-l-1} \frac{(-1)^{l+k}(m-k-1)!}{l!(n-l-i)!} \left\{ \begin{matrix} m-1 \\ m-k-1 \end{matrix} \right\} \binom{m-k-1}{i-k+1} \right) c_{n-i+1}. \end{aligned}$$

When  $\mu = 2$  in Theorem 4, we have for any integers  $n \geq 1$  and  $m \geq 2$ ,

$$\begin{aligned} & \underbrace{(c_0 + \cdots + c_0 + c_2)}_{m-1}^n + (m-1) \underbrace{(c_0 + \cdots + c_0 + c_1 + c_1)}_{m-2}^n \\ &= \frac{(n+2)!}{m!} \sum_{i=0}^{m-1} \left( \sum_{l=0}^{m-1} \sum_{k=0}^{m-l-1} \frac{(-1)^{l+k}(m-k-1)!}{l!(n-l-i)!} \left\{ \begin{matrix} m-1 \\ m-k-1 \end{matrix} \right\} \binom{m-k-1}{i-k+1} \right) c_{n-i+2}. \end{aligned}$$

## 4 Cauchy numbers of the second kind

The Cauchy numbers of the second kind  $\widehat{c}_n$  ( $n \geq 0$ ) are defined by

$$\widehat{c}_n = \int_0^1 (-x)(-x-1) \cdots (-x-n+1) dx$$

and the generating function of  $\widehat{c}_n$  is given by

$$\frac{x}{(1+x) \ln(1+x)} = \sum_{n=0}^{\infty} \widehat{c}_n \frac{x^n}{n!} \quad (|x| < 1)$$

([4, 11]). Several initial values are

$$\widehat{c}_0 = 1, \widehat{c}_1 = -\frac{1}{2}, \widehat{c}_2 = \frac{5}{6}, \widehat{c}_3 = -\frac{9}{4}, \widehat{c}_4 = \frac{251}{30}, \widehat{c}_5 = -\frac{475}{12}, \widehat{c}_6 = \frac{19087}{84}, \widehat{c}_7 = -\frac{36799}{24}.$$

In [10], an explicit expression of  $(\widehat{c}_l + \widehat{c}_m)^n$  for  $l, m, n \geq 0$  was determined, where with the classical umbral calculus notation (see, e.g., [12]),  $(\widehat{c}_l + \widehat{c}_m)^n$  is defined by

$$(\widehat{c}_l + \widehat{c}_m)^n := \sum_{j=0}^n \binom{n}{j} \widehat{c}_{l+j} \widehat{c}_{m+n-j}.$$



As special cases, we obtained

$$(\widehat{c}_0 + \widehat{c}_0)^n = n! \sum_{k=0}^n (-1)^{n-k} \frac{\widehat{c}_k}{k!} - n\widehat{c}_n, \quad (6)$$

$$(\widehat{c}_0 + \widehat{c}_1)^n = -\frac{(n+1)!}{2} \sum_{k=0}^n (-1)^{n-k} \frac{\widehat{c}_k}{k!} - \frac{1}{2}n\widehat{c}_{n+1}, \quad (7)$$

$$\begin{aligned} (\widehat{c}_0 + \widehat{c}_2)^n &= \frac{n!}{12} \sum_{k=0}^n \frac{(-1)^{n-k}}{k!} (2k(n+2k-2) + 5(n-k+2)(n-k+1)) \widehat{c}_k \\ &\quad - \frac{n}{2}\widehat{c}_{n+1} - \frac{n}{3}\widehat{c}_{n+2}, \end{aligned} \quad (8)$$

$$(\widehat{c}_1 + \widehat{c}_1)^n = \frac{n!}{12} \sum_{k=0}^n \frac{(-1)^{n-k}}{k!} ((n+1)(n+2) + k(8n-9k+19)) \widehat{c}_k - \widehat{c}_{n+1} - \frac{n+3}{6}\widehat{c}_{n+2} \quad (9)$$

(see [10]).

We shall consider the higher order recurrences for Cauchy numbers of the second kind:

$$(\widehat{c}_{l_1} + \cdots + \widehat{c}_{l_m})^n := \sum_{\substack{k_1 + \cdots + k_m = n \\ k_1, \dots, k_m \geq 0}} \frac{n!}{k_1! \cdots k_m!} \widehat{c}_{k_1+l_1} \cdots \widehat{c}_{k_m+l_m}.$$

As special cases, we shall have

$$\begin{aligned} &(\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_0)^n \\ &= \frac{n^2}{2}\widehat{c}_n + \frac{n!}{2} \sum_{i=0}^n \frac{(-1)^{n-i}(n-4i+2)}{i!} \widehat{c}_i \\ &= \frac{(n-1)(n-2)}{2}\widehat{c}_n + \frac{n!}{2} \sum_{i=0}^{n-1} \frac{(-1)^{n-i}(n-4i+2)}{i!} \widehat{c}_i, \end{aligned} \quad (10)$$

$$\begin{aligned} &(\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_1)^n \\ &= \frac{n(n-1)}{6}\widehat{c}_{n+1} - \frac{(n+1)!}{6} \sum_{i=0}^n \frac{(-1)^{n-i}(n-4i+3)}{i!} \widehat{c}_i. \end{aligned} \quad (11)$$

and

$$(\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_0 + \widehat{c}_0)^n = -\frac{n^3}{6}\widehat{c}_n + \frac{n!}{12} \sum_{i=0}^n \frac{(-1)^{n-i}(n^2 - 16in + 11n + 27i^2 - 33i + 12)}{i!} \widehat{c}_i, \quad (12)$$

$$(\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_0 + \widehat{c}_1)^n = -\frac{(n+1)^3}{24}\widehat{c}_{n+1} - \frac{(n+1)!}{48} \sum_{i=0}^{n+1} \frac{(-1)^{n-i}(n^2 - 16in + 13n + 27i^2 - 49i + 24)}{i!} \widehat{c}_i. \quad (13)$$

$\widehat{c}(x) = x/((1+x)\ln(1+x))$  satisfies the identity

$$\widehat{c}(x)^2 = -x\widehat{c}'(x) + \frac{1}{1+x}\widehat{c}(x). \quad (14)$$

Since for  $i, \nu \geq 0$  it holds that

$$x^i \widehat{c}^{(\nu)}(x) = \sum_{n=0}^{\infty} \frac{n!}{(n-i)!} \widehat{c}_{n+\nu-i} \frac{x^n}{n!}, \quad (15)$$

the identity (14) immediately leads to the formula

$$\sum_{k=0}^n \binom{n}{k} \widehat{c}_k \widehat{c}_{n-k} = -n\widehat{c}_n + n! \sum_{i=0}^n \frac{(-1)^{n-i} \widehat{c}_i}{i!} \quad (n \geq 0). \quad (16)$$

Differentiating both sides of (14) by  $x$  and dividing them by 2, we obtain

$$\widehat{c}(x)\widehat{c}'(x) = -\frac{1}{2}x\widehat{c}''(x) - \frac{x}{2(1+x)}\widehat{c}'(x) - \frac{1}{2(1+x)^2}\widehat{c}(x). \quad (17)$$

**Proposition 3.**

$$\widehat{c}(x)^3 = \frac{1}{2}x^2\widehat{c}''(x) + \frac{x(x-2)}{2(1+x)}\widehat{c}'(x) + \frac{x+2}{2(1+x)^2}\widehat{c}(x). \quad (18)$$

*Proof.* The proof is similat to that of Proposition 5. By (14) and (17), we get the result.  $\square$

Applying (15) and Proposition 3, we have the result of the third order.

**Theorem 5.** For  $n \geq 0$  we have

$$(\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_0)^n = \frac{n^2}{2} \widehat{c}_n + \frac{n!}{2} \sum_{i=0}^n \frac{(-1)^{n-i}(n-4i+2)}{i!} \widehat{c}_i.$$

Similarly, for  $n \geq 0$  we have

$$(\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_0 + \widehat{c}_0)^n = -\frac{n^3}{6} \widehat{c}_n + \frac{n!}{12} \sum_{i=0}^n \frac{(-1)^{n-i}(n^2 - 16in + 11n + 27i^2 - 33i + 12)}{i!} \widehat{c}_i.$$

$$\begin{aligned} & (\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_0 + \widehat{c}_0 + \widehat{c}_0)^n \\ &= \frac{n^4}{24} \widehat{c}_n + \frac{n!}{144} \sum_{i=0}^n \frac{(-1)^{n-i}}{i!} (n^3 - 48in^2 + 39n^2 + 243i^2n \\ &\quad - 393in + 176n - 256i^3 + 564i^2 - 476i + 144) \widehat{c}_i. \end{aligned}$$

In general, we can state the following.

**Theorem 6.** For any integers  $n \geq 1$  and  $m \geq 2$ , we have

$$\begin{aligned} & \underbrace{(\widehat{c}_0 + \cdots + \widehat{c}_0)}_m^n \\ &= \frac{n!}{(m-1)!} \sum_{i=0}^{n-1} \left( \sum_{l=0}^i \sum_{k=0}^{\min\{n-i, m-l-1\}} \frac{(-1)^{n-i+l}(m-k-1)!}{(i-l)!l!} \left\{ \begin{matrix} m-1 \\ m-k-1 \end{matrix} \right\} \binom{n-i}{k} \right. \\ &\quad \left. \sum_{i=0}^{n-1} \sum_{k=0}^{m-1} \left\{ \begin{matrix} m-1 \\ k \end{matrix} \right\} \binom{n-i-1}{m-k-1} \frac{(-1)^{n-i+k-1}}{(i-k)!} \right) \widehat{c}_i \\ &\quad + \sum_{l=0}^{\min\{n, m\}} (-1)^l \binom{n}{l} \widehat{c}_n. \end{aligned}$$

**Theorem 7.** For any integers  $n \geq 1$  and  $m \geq 2$ , we have

$$\begin{aligned}
& \sum_{\substack{\kappa_1 + \dots + \kappa_m = \mu \\ \kappa_1, \dots, \kappa_m \geq 0}} \frac{\mu!}{\kappa_1! \dots \kappa_m!} (\widehat{c}_{\kappa_1} + \dots + \widehat{c}_{\kappa_m})^n \\
&= \frac{(n + \mu)!}{(m - 1)!} \sum_{i=0}^{n+\mu-1} \left( \sum_{l=0}^i \sum_{k=0}^{\min\{n+\mu-i, m-l-1\}} \frac{(-1)^{n+\mu-i+l} (m-k-1)!}{(i-l)! l!} \left\{ \begin{matrix} m-1 \\ m-k-1 \end{matrix} \right\} \binom{n+\mu-i}{k} \right) \\
&\quad \sum_{i=0}^{n+\mu-1} \sum_{k=0}^{m-1} \left\{ \begin{matrix} m-1 \\ k \end{matrix} \right\} \binom{n+\mu-i-1}{m-k-1} \frac{(-1)^{n+\mu-i+k-1}}{(i-k)!} \widehat{c}_i \\
&\quad + \sum_{l=0}^{\min\{n+\mu, m\}} (-1)^l \binom{n+\mu}{l} \widehat{c}_{n+\mu}.
\end{aligned}$$

If  $m = 4$  and  $\mu = 2$  in Theorem 7, we have

$$\begin{aligned}
& 12(\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_1 + \widehat{c}_1)^n + 4(\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_0 + \widehat{c}_2)^n \\
&= -\binom{n+1}{3} \widehat{c}_{n+2} + \frac{(n+2)!}{12} \sum_{l=0}^{n+1} \frac{(-1)^{n-l}}{l!} (27l^2 - 16nl - 65l + n^2 + 15n + 38) \widehat{c}_l.
\end{aligned}$$

If  $m = 4$  and  $\mu = 3$  in Theorem 7, we have

$$\begin{aligned}
& 24(\widehat{c}_0 + \widehat{c}_1 + \widehat{c}_1 + \widehat{c}_1)^n + 36(\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_1 + \widehat{c}_2)^n + 4(\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_0 + \widehat{c}_3)^n \\
&= -\binom{n+2}{3} \widehat{c}_{n+3} + \frac{(n+3)!}{12} \sum_{l=0}^{n+2} \frac{(-1)^{n-l+1}}{l!} (27l^2 - 16nl - 81l + n^2 + 17n + 54) \widehat{c}_l.
\end{aligned}$$

If  $m = 5$  and  $\mu = 1$  in Theorem 7, we have

$$\begin{aligned}
& 5(\widehat{c}_0 + \widehat{c}_0 + \widehat{c}_0 + \widehat{c}_0 + \widehat{c}_1)^n \\
&= \binom{n}{4} \widehat{c}_{n+1} \\
&\quad - \frac{(n+1)!}{144} \sum_{l=0}^n \frac{(-1)^{n-l+1}}{l!} (256l^3 - (243n + 807)l^2 \\
&\quad + (48n^2 + 489n + 917)l - (n^3 + 42n^2 + 257n + 360)) \widehat{c}_l.
\end{aligned}$$

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School of Mathematics and Statistics  
Wuhan University  
Wuhan 430072  
CHINA  
E-mail address: komatsu@whu.edu.cn